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TRANSIENT RESPONSE OF A MOVING SPHERICAL SHELL IN AN ACOUSTIC MEDIUM

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Abstract—A ring-stiffened spherical shell is submerged in an acoustic medium. The shell is thin and elastic. The acoustic medium is inviscid, irrotational and compressible. The center of mass of the shell is subjected to a translational acceleration which is an arbitrary function of time. The absolute displacements of the shell are expressed in terms of the relative displacements and the displacement of the base of the shell, base being defined as the rigid ring placed at the equator. The motion of the acoustic medium is governed by the wave equation. The transient response of the shell is investigated numerically. The results are compared with the results of the in-vacuo response. The effects of the plane wave approximation and the base velocity on the transient response of the shell are studied. The numerical results show that the plane wave approximation accurately predicts the response of the shell in the acoustic medium for short times after excitation. The displacements of the shell in fluid are larger than those in vacuo. But when the base of the shell is restrained from translating, the displacements in fluid are smaller than those in vacuo. Therefore, base translation has a very significant effect on the transient response of the shells submerged in an acoustic medium.

1. INTRODUCTION

Dynamics of shells submerged in an acoustic medium has been the subject of many investigations. The steady-state and transient responses of submerged cylindrical shells[1–3, 5–9, 11] and spherical shells[1, 4, 8, 10, 12, 13] have been studied quite extensively. In most of these studies, the resultant applied force is zero and, hence, the center of mass of the shell does not have any acceleration or velocity. However, in [11] and [13] the shell is acted upon by a concentrated force at the apex. Accordingly, there is a net resultant force on the shell which imparts an acceleration to the center of mass of the shell. This acceleration may be constant or a function of time, depending on the type of the applied force. So the shell starts moving in the acoustic medium. In [11] and [13], this part of the problem is not considered. As a matter of fact, to the author's knowledge, the transient response of a spherical shell moving in an acoustic medium has not been studied at all. It is one of the purposes of this paper to study the problem presented above.

The problem to be considered is that of a ring-stiffened spherical shell submerged in an acoustic medium. The center of mass of the shell is subjected to a translational acceleration which is an arbitrary function of time. The equations governing the transient response of this three-dimensional shell-fluid interaction problem will be derived using the concept of relative motion. The effect of the plane wave approximation will be studied. To illustrate the effect of the acoustic medium on the response of the shell, its *in vacuo* response will be obtained. Finally, the ring-stiffner will be held fixed and the effect of the velocity of the center of mass of the shell on the transient response will be investigated.

2. THEORY

A ring-stiffened spherical shell is submerged in an acoustic medium (Fig. 1). The rigid ring-stiffener is placed at the equator of the shell, which will be called the base from now on. The shell is thin and the shell material is linearly elastic, homogeneous and isotropic. The acoustic fluid is inviscid, irrotational and compressible. It is assumed that, due to some external source, the center of mass, hence the base, of the shell is subjected to an acceleration which may be any function of time. The transient response of the shell will be investigated. It is assumed from the beginning that motions are small.

The equations governing the behavior of the shell are given by Flügge[14] and they are, in nondimensional form,

$$[L]\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix} = (1-\nu^2)\sin\phi \begin{Bmatrix} \partial^2\bar{u}/\partial\tau^2 \\ \partial^2\bar{v}/\partial\tau^2 \\ q_r - \partial^2\bar{w}/\partial\tau^2 \end{Bmatrix}, \qquad (1)$$

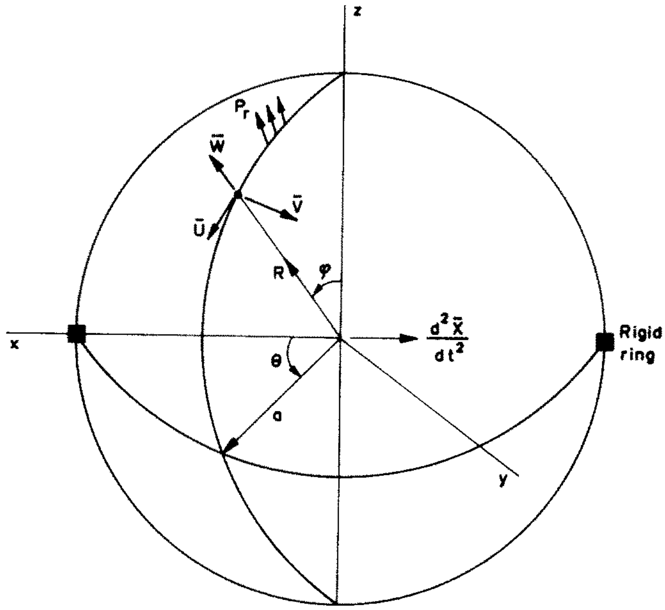

Fig. 1. Geometry of the problem.

in which $[L]$ is a 3×3 linear matrix differential operator whose elements are given in the Appendix. The motion of an inviscid, irrotational and compressible fluid undergoing small oscillations is governed by the wave equation. In spherical coordinates its nondimensional form is

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\cot\phi}{r^2}\frac{\partial\Phi}{\partial\phi}+\frac{1}{r^2\sin^2\phi}\frac{\partial^2\Phi}{\partial\theta^2}=\frac{1}{s^2}\frac{\partial^2\Phi}{\partial\tau^2}. \tag{2}$$

The nondimensional quantities appearing in eqns (1) and (2) are related to the corresponding physical quantities through the following relations:

$$\left.\begin{aligned}&\bar{u}=\frac{\bar{U}}{a},\quad \bar{v}=\frac{\bar{V}}{a},\quad \bar{w}=\frac{\bar{W}}{a},\quad \Phi=\frac{\bar{\Phi}}{ac_s},\quad k=\frac{1}{12}\left(\frac{h}{a}\right)^2,\\ &\tau=\frac{c_s}{a}t,\quad r=\frac{R}{a},\quad q_r=\frac{a}{Eh}p_r,\quad s=\frac{c_f}{c_s},\quad c_s^2=\frac{E}{\rho_s}\end{aligned}\right\} \tag{3}$$

In the foregoing expressions a and h are, respectively, the radius and the thickness of the shell. $\bar{U}$, $\bar{V}$, $\bar{W}$ are the absolute meridional, circumferential and radial displacement components, respectively. The mass density and the modulus of elasticity of the shell material are denoted by ρ_s and E, respectively. The physical time is t and p_r is the load in radial direction. Poisson's ratio is denoted by ν, c_f is the speed of sound in the fluid, and the velocity potential is $\bar{\Phi}$. Finally, ϕ, θ and R are meridional, cricumferential and radial coordinates, respectively, as shown in Fig. 1.

For a shell accelerating as shown in Fig. 1, the absolute and relative displacements of a point on the shell are related through

$$\begin{Bmatrix}\bar{u}\\ \bar{v}\\ \bar{w}\end{Bmatrix}=\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}+X\begin{Bmatrix}-\cos\theta\cos\phi\\ \sin\theta\\ -\cos\theta\sin\phi\end{Bmatrix}, \tag{4}$$

in which

$$u=\frac{U}{a},\quad v=\frac{V}{a},\quad w=\frac{W}{a},\quad X=\frac{\bar{X}}{a}. \tag{5}$$

Here, U, V, W are the displacement components relative to the base and $\bar{X}$ is the displacement of the base of the shell.

Substituting eqns (4) into eqns (1) one obtains

$$[L]\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = (1-\nu^2)\sin\phi \begin{Bmatrix} \ddot{u} - \cos\theta\cos\phi\,\ddot{X} \\ \ddot{v} + \sin\theta\ddot{X} \\ q_r - \ddot{w} + \cos\theta\sin\phi\ddot{X} \end{Bmatrix} \tag{6}$$

because

$$[L]\begin{Bmatrix} -\cos\theta\cos\phi \\ \sin\theta \\ -\cos\theta\sin\phi \end{Bmatrix} = 0. \tag{7}$$

For the present problem the incident pressure is zero. Therefore,

$$q_r = -q_f, \tag{8}$$

in which q_f is the nondimensional hydrodynamic fluid pressure. In dimensional form, the hydroynamic fluid pressure p_f is given by

$$p_f = -\rho_f \frac{\partial\bar{\Phi}}{\partial t}, \tag{9}$$

accordingly,

$$q_f = -f\dot{\Phi} \quad \text{at} \quad r = 1, \tag{10}$$

in which $f = (a\rho_f)/(h\rho_s)$ is called the fluid-shell interaction parameter and the dots in eqns (6) and (10) denote differentiation with respect to nondimensional time τ. Moreover, ρ_f is the mass density of the fluid.

The kinematic boundary condition states that the fluid at a point next to the shell must have the same velocity in the direction of the normal to the shell as the shell itself does at point. In dimensional form,

$$\frac{\partial\bar{W}}{\partial t} = \frac{\partial\bar{\Phi}}{\partial R} \quad \text{at} \quad R = a. \tag{11}$$

Nondimensionalizing and using eqn (4) one obtains

$$\frac{\partial\Phi}{\partial r} = \dot{w} - \cos\theta\sin\phi\dot{X} \quad \text{at} \quad r = 1. \tag{12}$$

Since the loading is symmetric with respect to the $x-z$ plane of Fig. 1, the response of the shell will also be symmetric with respect to the same plane. Therefore, one can let

$$(u, v, w, \Phi) = \sum_{n=0}^{\infty} (u_n \cos n\theta, v_n \sin n\theta, w_n \cos n\theta, \Phi_n \cos n\theta), \tag{13}$$

in which n is the circumferential mode number. It should be emphasized that such an expansion is acceptable if significant nonlinear effects are not present.

Substituting eqns (8), (10) and (13) into eqns (6) and (2) one obtains

$$[L_n]\begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} = (1-\nu^2)\sin\phi \begin{Bmatrix} \ddot{u}_n - \delta_1\ddot{X}\cos\phi \\ \ddot{v}_n + \delta_1\ddot{X} \\ -\ddot{w}_n + \delta_1\ddot{X}\sin\phi + f\dot{\Phi}_n(r=1) \end{Bmatrix} \tag{14}$$

$$\frac{\partial^2\Phi_n}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi_n}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi_n}{\partial\phi^2} + \frac{\cot\phi}{r^2}\frac{\partial\Phi_n}{\partial\phi} - \frac{n^2\Phi_n}{r^2\sin^2\phi} = \frac{1}{s^2}\ddot{\Phi}_n, \tag{15}$$

in which

$$\delta_1 = \begin{cases} 1 & \text{for} \quad n = 1 \\ 0 & \text{for} \quad n \neq 1. \end{cases} \tag{16}$$

The kinematic boundary condition (12) takes the following form:

$$\frac{\partial \Phi_n}{\partial r} = \dot{w}_n - \delta_1 \dot{X} \sin \phi \quad \text{at} \quad r = 1. \tag{17}$$

In eqn (14), the matrix differential operator $[L_n]$ is obtained from $[L]$ by substituting correspondings n's for the derivatives with respect to the circumferential coordinate θ.

The boundary, apex and initial conditions for the present problem are all homogeneous and they will be given later in the paper. Accordingly, from eqns (14)–(17) it is seen that the translational acceleration of the shell excites only the cantilever beam mode, $n = 1$. For $n \neq 1$ the problem is a free-vibration problem, because there is no load term on the right hand side of the equations.

To be able to get rid of the third and fourth order derivatives in u_n and w_n in eqs (14), a new variable is defined as follows:

$$m_n = \frac{\partial^2 w_n}{\partial \phi^2} - \frac{\partial u_n}{\partial \phi}. \tag{18}$$

For the present problem, since only the $n = 1$ mode is excited, the subscript n will be discarded from the equations which take the following final form:

$$[L^*] \begin{Bmatrix} u \\ v \\ w \\ m \end{Bmatrix} = (1 - \nu^2) \sin \phi \begin{Bmatrix} \ddot{u} - \ddot{X} \cos \phi \\ \ddot{v} + \ddot{X} \\ -\ddot{w} + \ddot{X} \sin \phi + f\dot{\Phi}(r = 1) \\ 0 \end{Bmatrix} \tag{19}$$

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\cot \phi}{r^2}\frac{\partial \Phi}{\partial \phi} - \frac{\Phi}{r^2 \sin^2 \phi} = \frac{1}{s^2}\ddot{\Phi}, \tag{20}$$

$$\frac{\partial \Phi}{\partial r} = \dot{w} - \dot{X} \sin \phi \quad \text{at} \quad r = 1, \tag{21}$$

in which $[L^*]$ is obtained from $[L_n]$ using the relation (18) and it is a 4×4 operator.

The transient response of a ring-stiffened spherical shell subjected to a base acceleration is governed by eqns (19)–(21). The problem is completed by specifying the apex, boundary and initial conditions.

The apex conditions at $\phi = 0°$ are given in [15–17], and they are

$$\frac{\partial u}{\partial \phi} = u + v = w = m = \Phi = 0. \tag{22}$$

Since the ring-stiffener is assumed to be rigid, the boundary conditions at $\phi = \pi/2$ are

$$u = v = w = \frac{\partial w}{\partial \phi} = \frac{\partial \Phi}{\partial \phi} = 0. \tag{23}$$

The initial conditions at $\tau = 0$ are

$$u = \dot{u} = v = \dot{v} = w = \dot{w} = \Phi = \dot{\Phi} = 0. \tag{24}$$

Finally, it is assumed that

$$\frac{\partial \Phi}{\partial r} \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \tag{25}$$

3. NUMERICAL PROCEDURE

Equations (19)–(25), governing the present problem, are solved numerically by finite difference techniques. The meridional and radial derivatives appearing in the equations are replaced by the conventional central finite difference approximations. However, at the apex of the shell the forward finite difference approximations are used. Time derivatives are approximated by Houbolt's[18] backward differencing scheme. Using the finite difference approximations mentioned, the governing equations are reduced to sets of albegraic equations which are, then solved using Potters'[19] form of Gaussian elimination. This method is described in [15] in detail for one-dimensional problems. For two-dimensional problems, as in the present case, the method is similar.

For the difference approximations the shell meridian is divided into eighteen equal increments; in other words, the meridional increment $\Delta\phi$ is taken to be five degrees. The nondimensional mesh size in the radial direction is $\Delta r = 0.25$, and the number of equal increments in this direction is eleven. The finite difference scheme is shown in Fig. 2. The physical time increment Δt is taken to be 0.1 msec. A computer program, named SØFLIP-1, was written in FØRTRAN language. Each run, with a response time of 1.6 msec, took about a total of 8 min of the IBM 370/145 computer available. It was necessary to use double-precision throughout. The plane wave approximation, which will be explained in the following section, took only 1 min for the same response time.

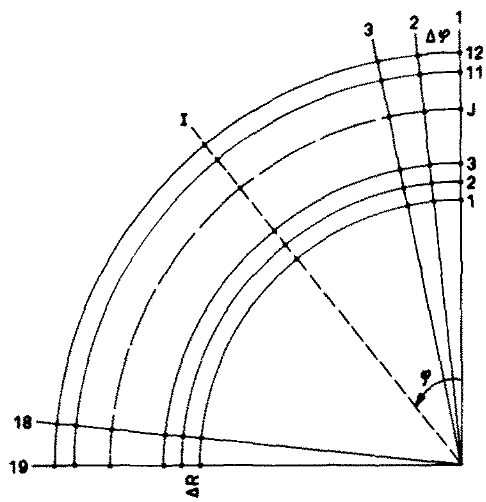

Fig. 2. Finite difference scheme.

To the author's knowledge, there is no information available at present which provides direct numerical verification of the solution of the present problem. To establish the validity of the method, some simpler cases were examined. A second program was written for the same problem, but now *in vacuo*. The density of the acoustic medium was allowed to go to zero in SØFLIP-1, and the results coincided with those obtained from the second program. A third program was written for the transient response analysis of a rigid sphere in an acoustic medium. The steady-state responses of the rigid sphere obtained from the third program coincided with the analytical results of Lamb[20]. Then, the elastic modulus of the shell material was taken to be very large in SØFLIP-1. The *transient* response for this almost rigid shell coincided with the transient response of the rigid sphere for the same acceleration function.

It is known that the $\phi - r$ grid size diverges as r increases, when one tries to solve an exterior problem in polar coordinates using the finite-difference scheme shown in Fig. 2. Moreover, the finite difference domain needs to be increased as time increases. But, since only very short times are dealt with in this work (Maximum response time is 1.6 msec, to be exact), the distance that the waves have travelled at the end of the response time is not very large. The distance that the waves have reached at 1.6 msec is still within the domain considered in this work.

As an additional check for the accuracy of the numerical scheme, the nondimensional mesh size in radial direction was increased to $\Delta r = 0.50$ with eleven equal increments in this direction. So a larger domain was covered for this exterior problem. Then, Δr was reduced to 0.125 but the number of equal increments was increased to twentytwo. Hence, the domain, although the same as that for $\Delta r = 0.25$, was divided into finer meshes. The differences among the numerical results obtained using these three different sets of mesh size and number of increments were within the limits acceptable for numerical solutions.

The accuracy of the numerical scheme used in this work can also be checked via an indirect approach. As will be seen in the following section, the response of the shell considered is of long periodicity. Therefore, the damping provided by the plane wave approximation is insignificant at early time for this problem. Accordingly, the plane wave solution should essentially coincide with the exact solution for short times after excitation. It will be seen later in the paper that they actually do coincide. Moreover, we do not have the problem of divergence of the $\phi - r$ grid size for the plane wave solution. The details of the numerical solution will be presented in a later work.

4. RESULTS AND CONCLUSIONS

A steel shell is immersed in water and its center of mass is subjected to the acceleration shown in Fig. 3. The amplitude of the acceleration is 100 cm/sec², but it is irrelevant because the problem solved is linear. The velocity of the base is obtained by integrating the acceleration function using the trapezoidal rule. The duration of the base acceleration is 0.4 msec. The physical dimensions and properties used are as follows:

$$a = 100\,\text{cm}, \quad h = 2\,\text{cm}, \quad (a/h = 50), \quad E = 2109240\,\text{kg/cm}^2, \quad \nu = 1/3,$$

$$\rho_s = 8.006 \times 10^{-6}\,\text{kg-sec}^2/\text{cm}^4, \; \rho_f = 1.000 \times 10^{-6}\,\text{kg-sec}^2/\text{cm}^4, \quad c_f = 152\ 000\,\text{cm/sec}.$$

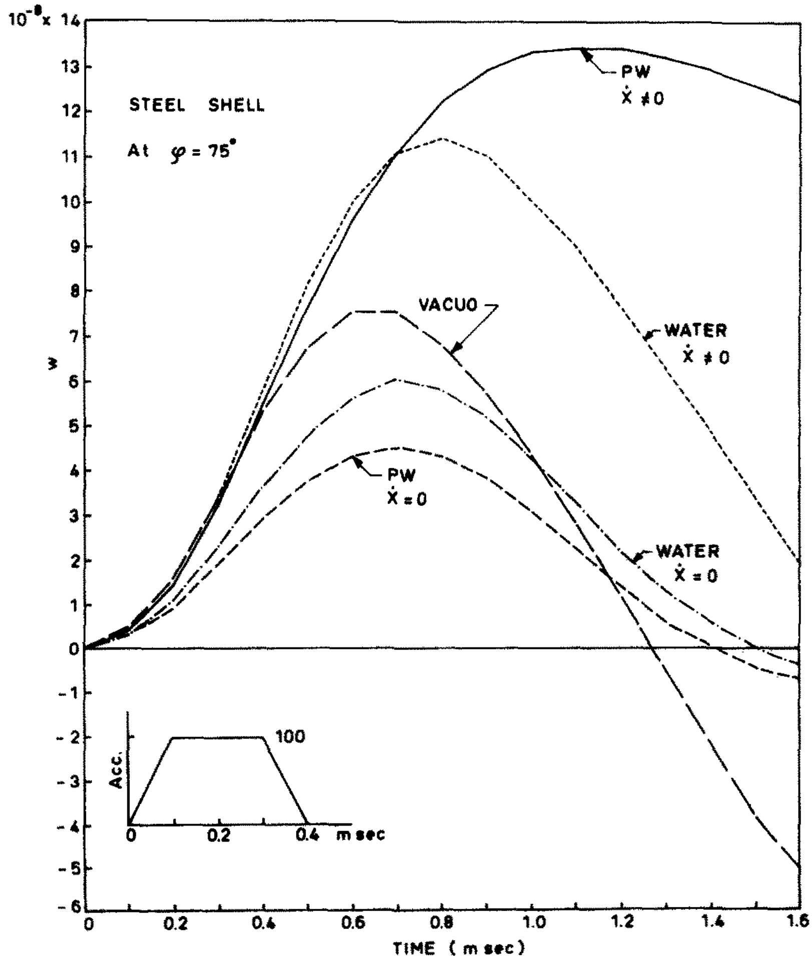

Fig. 3. Nondimensional radial displacement vs time (Maximum acceleration is 100 cm/sec²).

The plane wave approximation is, in dimensional form, expreseed by the following reduced wave equation:

$$\frac{\partial \bar{\Phi}}{\partial R} = -\frac{1}{c_f}\frac{\partial \bar{\Phi}}{\partial t}. \tag{26}$$

For $n = 1$, in nondimensional form, it is

$$\frac{\partial \Phi}{\partial r} = -\frac{1}{s}\dot{\Phi}. \tag{27}$$

Using eqn (21), $f\dot{\Phi}(r = 1)$ term in eqn (19) can be replaced by

$$f\dot{\Phi} = -fs(\dot{w} - \dot{X}\sin\phi) \quad \text{at} \quad r = 1. \tag{28}$$

So the shell eqns (19) can be uncoupled from the wave eqn (20).

The nondimensional radial displacement w at $\phi = 75°$ is plotted as a function of time in Fig. 3. The radial displacement at this point is a representative of those at other points, and, moreover, this

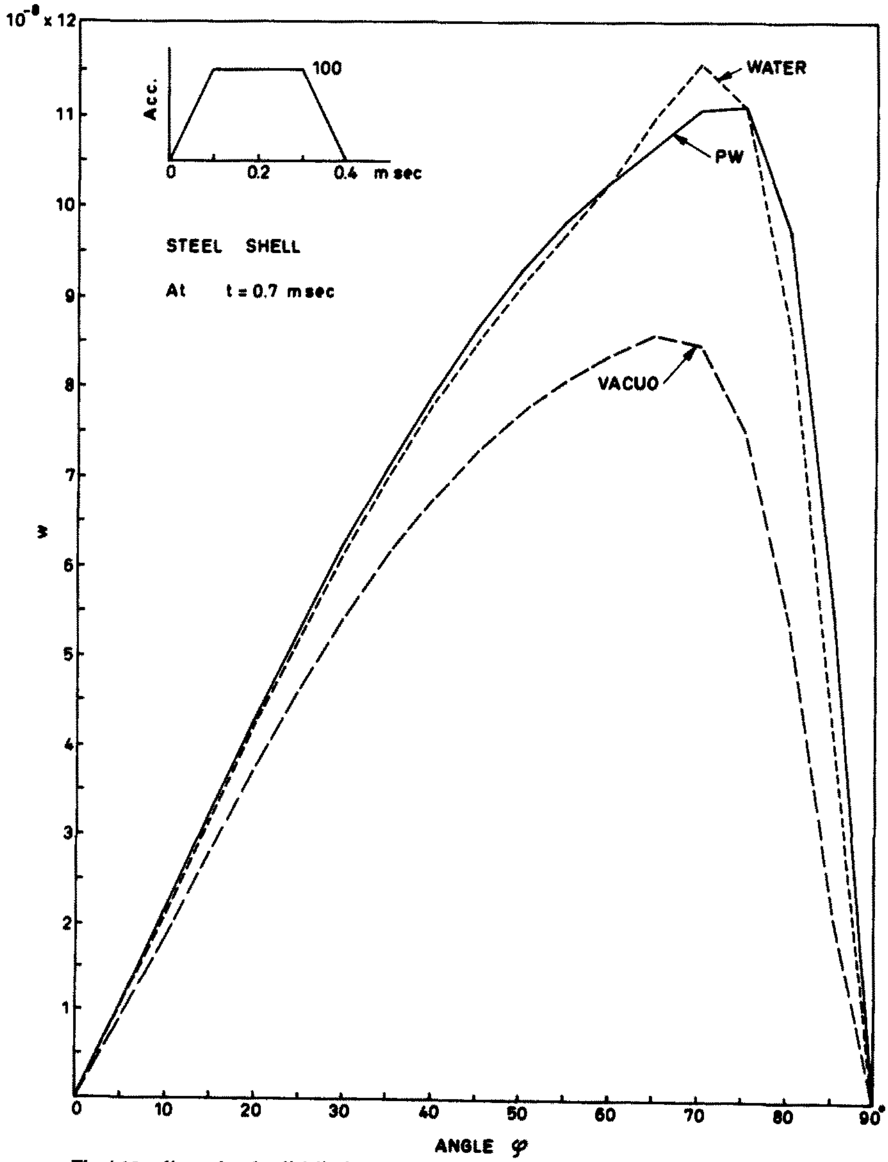

Fig. 4. Nondimensional radial displacement vs angle. (Maximum acceleration is 100 cm/sec^2.)

point has the maximum radial displacement at a greater number of time stations. There are five curves in Fig. 3, and the label PW stands for the plane wave approximation. To see the effect of the velocity of the base on the transient response, in the program $\dot{X}$ is set equal to zero, keeping $\ddot{X}$ the same as before. This corresponds to the case in which the shell is loaded by the inertial forces $(-\cos\theta\cos\phi\ddot{X})$, $(\sin\theta\ddot{X})$ and $(\cos\theta\sin\phi\ddot{X})$ in meridional, circumferential and radial directions, respectively, as equivalent surface loads.

An investigation of Fig. 3 reveals the following conclusions. The response in water is undamped for the time range considered. A similar conclusion was obtained by Lyons *et al.*[6] for their problem for $n = 1$. The period of oscillation of the shell in water is greater than the period of the vacuum response. This is, again, in agreement with the results of [6]. The maximum amplitude in water is about 1.5 times greater than that in vacuo. The presence of the acoustic medium increases the period and the amplitude when the base velocity is considered in the analysis. Holding the ring-stiffner fixed reduces the amplitude to a value less than that in vacuo, but the period is still slightly larger. The plane wave approximation predicts the transient response accurately for short times after excitation and the response obtained from the plane wave approximation is apparently damped. The maximum amplitude predicted by the plane wave approximation is about 20% larger than that of the exact solution. However, the occurence time of the maximum amplitude is not

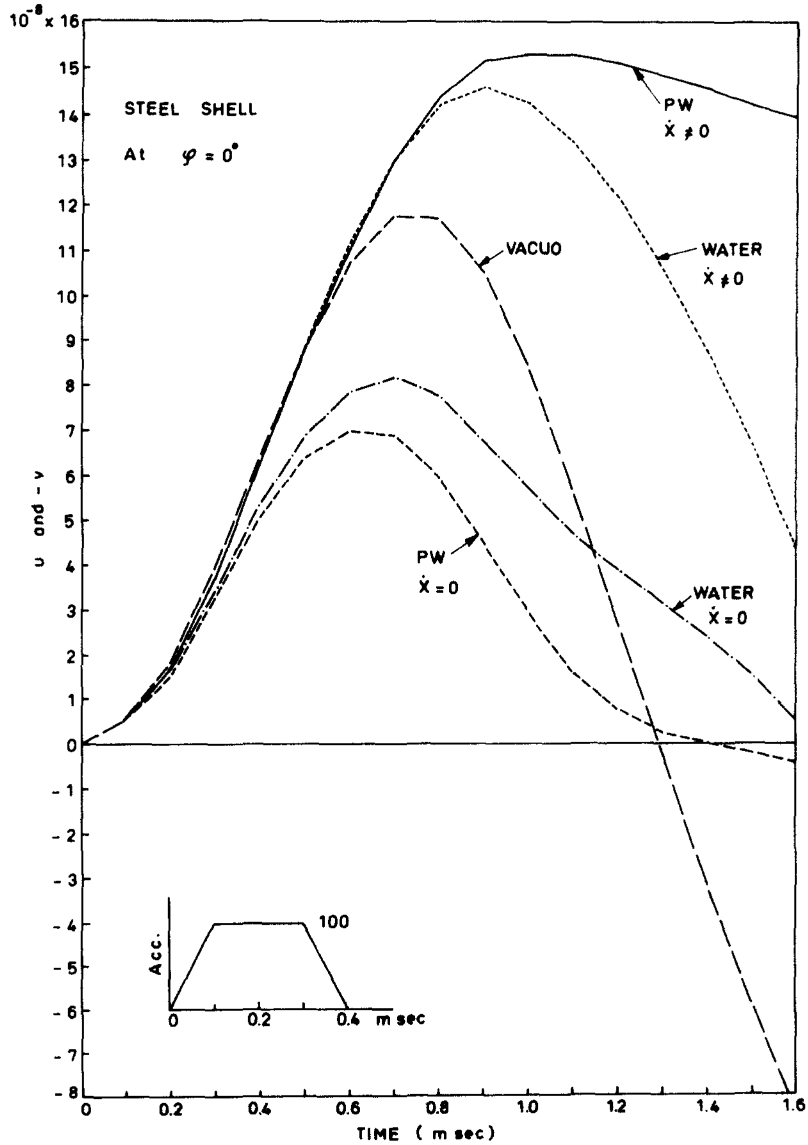

Fig. 5. Nondimensional meridional and circumferential displacements vs time. (Maximum acceleration is 100 cm/sec^2.)

predicted accurately. The shell oscillates about its equilibrium position ($w = 0$) both in vacuo and in water. To see if the phenomenon of beating takes place, it is necessary to use a longer response time[13].

The results obtained from the plane wave approximation and presented in Fig. 3 are not unexpected. It is well known that the plane wave approximation overestimates the fluid resistance. Accordingly, when $\dot{X} = 0$ and the shell is thus loaded with the given inertial forces, the above-mentioned overestimation in the fluid resistance decreases the shell response. On the other hand, when the shell is allowed to move in the acoustic medium ($\dot{X} \neq 0$), the very fact that the plane wave approximation predicts too large a resistive force results in an overestimation of the response amplitudes and a very significant decrease in the oscillatory nature of the response. Similar conclusions are obtained from the results of the exact acoustic field equation.

In Fig. 4, the nondimensional radial displacement is plotted as a function of the meridional coordinate for the shell in vacuo, for the shell in water and also for the plane wave approximation. The curves are for $t = 0.7$ msec beyond which the plane wave approximation starts giving increasingly inaccurate results.

The nondimensional meridional and circumferential displacements at the apex of the shell are plotted in Fig. 5. The conclusions obtained from this figure are similar to the ones obtained from Fig. 3. However, now the maximum amplitude and its occurrence time predicted by the plane wave approximation are more accurate.

The most important conclusions of the present study can be summarized as follows:

1. When it is assumed that the velocity of the center of mass of the shell is zero, the presence of an acoustic medium reduces the amplitude but increases the period.

2. When the velocity is considered, the presence of the acoustic medium increases the amplitude and also the period significantly.

3. When the velocity is considered, the plane wave approximation predicts the response accurately for short times after excitation.

4. The fluid resistance decreases the amplitude of the vibrations of the shell held fixed in an acoustic medium. On the other hand, it is also this resistance which increases the amplitude if the shell moves in the medium.

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APPENDIX

L_{11}: $(1+k)\left(\sin\varphi\frac{\partial^2}{\partial\varphi^2}+\cos\varphi\frac{\partial}{\partial\varphi}-\frac{\cos^2\varphi+\nu\sin^2\varphi}{\sin\varphi}+\frac{1-\nu}{2\sin\varphi}\frac{\partial^2}{\partial\theta^2}\right)$

L_{12}: $(1+k)\left(\frac{1+\nu}{2}\frac{\partial^2}{\partial\theta\partial\varphi}-\frac{3-\nu}{2}\cot\varphi\frac{\partial}{\partial\theta}\right)$

L_{13}: $(1+k)(1+\nu)\sin\varphi\frac{\partial}{\partial\varphi}-k\left[\sin\varphi\frac{\partial^3}{\partial\varphi^3}+\cos\varphi\frac{\partial^2}{\partial\varphi^2}+(1-\cot^2\varphi)\sin\varphi\frac{\partial}{\partial\varphi}+\frac{1}{\sin\varphi}\frac{\partial^3}{\partial\theta^2\partial\varphi}-\frac{2\cos\varphi}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right]$

L_{21}: $(1+k)\frac{(1+\nu)}{2}\frac{\partial^2}{\partial\theta\partial\varphi}+(1+k)\frac{(3-\nu)}{2}\cot\varphi\frac{\partial}{\partial\theta}$

L_{22}: $(1+k)\left\{\frac{(1-\nu)}{2}\left[\sin\varphi\frac{\partial^2}{\partial\varphi^2}+\cos\varphi\frac{\partial}{\partial\varphi}-\sin\varphi\,(\cot^2\varphi-1)\right]+\frac{1}{\sin\varphi}\frac{\partial^2}{\partial\theta^2}\right\}$

L_{23}: $(1+k)(1+\nu)\frac{\partial}{\partial\theta}-k\left(\frac{\partial^3}{\partial\theta\partial\varphi^2}+\cot\varphi\frac{\partial^2}{\partial\theta\partial\varphi}+2\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\varphi}\frac{\partial^3}{\partial\theta^3}\right)$

L_{31}: $(1+k)(1+\nu)\left(\sin\varphi\frac{\partial}{\partial\varphi}+\cos\varphi\right)-k\left[\sin\varphi\frac{\partial^3}{\partial\varphi^3}+2\cos\varphi\frac{\partial^2}{\partial\varphi^2}-\frac{\cos^2\varphi}{\sin\varphi}\frac{\partial}{\partial\varphi}\right.$

$\left.+(3+\cot^2\varphi)\cos\varphi+\frac{1}{\sin\varphi}\frac{\partial^3}{\partial\varphi\partial\theta^2}+\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right]$

L_{32}: $(1+k)(1+\nu)\frac{\partial}{\partial\theta}-k\left[\frac{\partial^3}{\partial\theta\partial\varphi^2}-\cot\varphi\frac{\partial^2}{\partial\theta\partial\varphi}+(3+\cot^2\varphi)\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\varphi}\frac{\partial^3}{\partial\theta^3}\right]$

L_{33}: $2(1+k)(1+\nu)\sin\varphi+k\left[\sin\varphi\frac{\partial^4}{\partial\varphi^4}+2\cos\varphi\frac{\partial^3}{\partial\varphi^3}-(1+\nu+\cot^2\varphi)\sin\varphi\frac{\partial^2}{\partial\varphi^2}\right.$

$+(2-\nu+\cot^2\varphi)\cos\varphi\frac{\partial}{\partial\varphi}-2(1+\nu)\sin\varphi+\frac{2}{\sin\varphi}\frac{\partial^4}{\partial\theta^2\partial\varphi^2}-2\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial^3}{\partial\theta^2\partial\varphi}$

$\left.+\frac{(3-\nu+4\cot^2\varphi)}{\sin\varphi}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\sin^3\varphi}\frac{\partial^4}{\partial\theta^4}\right].$